Pareto Distributions Second Edition

Barry C. Arnold



Pareto Distributions

Second Edition

MONOGRAPHS ON STATISTICS AND APPLIED PROBABILITY

General Editors

F. Bunea, V. Isham, N. Keiding, T. Louis, R. L. Smith, and H. Tong

- 1. Stochastic Population Models in Ecology and Epidemiology M.S. Barlett (1960)
- 2. Queues D.R. Cox and W.L. Smith (1961)
- 3. Monte Carlo Methods J.M. Hammersley and D.C. Handscomb (1964)
- 4. The Statistical Analysis of Series of Events D.R. Cox and P.A.W. Lewis (1966)
- 5. Population Genetics W.J. Ewens (1969)
- 6. Probability, Statistics and Time M.S. Barlett (1975)
- 7. Statistical Inference S.D. Silvey (1975)
- 8. The Analysis of Contingency Tables B.S. Everitt (1977)
- 9. Multivariate Analysis in Behavioural Research A.E. Maxwell (1977)
- 10. Stochastic Abundance Models S. Engen (1978)
- 11. Some Basic Theory for Statistical Inference E.J.G. Pitman (1979)
- 12. Point Processes D.R. Cox and V. Isham (1980)
- 13. Identification of Outliers D.M. Hawkins (1980)
- 14. Optimal Design S.D. Silvey (1980)
- 15. Finite Mixture Distributions B.S. Everitt and D.J. Hand (1981)
- 16. Classification A.D. Gordon (1981)
- 17. Distribution-Free Statistical Methods, 2nd edition J.S. Maritz (1995)
- 18. Residuals and Influence in Regression R.D. Cook and S. Weisberg (1982)
- 19. Applications of Queueing Theory, 2nd edition G.F. Newell (1982)
- 20. Risk Theory, 3rd edition R.E. Beard, T. Pentikäinen and E. Pesonen (1984)
- 21. Analysis of Survival Data D.R. Cox and D. Oakes (1984)
- 22. An Introduction to Latent Variable Models B.S. Everitt (1984)
- 23. Bandit Problems D.A. Berry and B. Fristedt (1985)
- 24. Stochastic Modelling and Control M.H.A. Davis and R. Vinter (1985)
- 25. The Statistical Analysis of Composition Data J. Aitchison (1986)
- 26. Density Estimation for Statistics and Data Analysis B.W. Silverman (1986)
- 27. Regression Analysis with Applications G.B. Wetherill (1986)
- 28. Sequential Methods in Statistics, 3rd edition G.B. Wetherill and K.D. Glazebrook (1986)
- 29. Tensor Methods in Statistics P. McCullagh (1987)
- 30. Transformation and Weighting in Regression R.J. Carroll and D. Ruppert (1988)
- 31. Asymptotic Techniques for Use in Statistics O.E. Bandorff-Nielsen and D.R. Cox (1989)
- 32. Analysis of Binary Data, 2nd edition D.R. Cox and E.J. Snell (1989)
- 33. Analysis of Infectious Disease Data N.G. Becker (1989)
- 34. Design and Analysis of Cross-Over Trials B. Jones and M.G. Kenward (1989)
- 35. Empirical Bayes Methods, 2nd edition J.S. Maritz and T. Lwin (1989)
- 36. Symmetric Multivariate and Related Distributions K.T. Fang, S. Kotz and K.W. Ng (1990)
- 37. Generalized Linear Models, 2nd edition P. McCullagh and J.A. Nelder (1989)
- 38. Cyclic and Computer Generated Designs, 2nd edition J.A. John and E.R. Williams (1995)
- 39. Analog Estimation Methods in Econometrics C.F. Manski (1988)
- 40. Subset Selection in Regression A.J. Miller (1990)
- 41. Analysis of Repeated Measures M.J. Crowder and D.J. Hand (1990)
- 42. Statistical Reasoning with Imprecise Probabilities *P. Walley* (1991)
- 43. Generalized Additive Models T.J. Hastie and R.J. Tibshirani (1990)
- 44. Inspection Errors for Attributes in Quality Control N.L. Johnson, S. Kotz and X. Wu (1991)
- 45. The Analysis of Contingency Tables, 2nd edition B.S. Everitt (1992)
- 46. The Analysis of Quantal Response Data B.J.T. Morgan (1992)
- 47. Longitudinal Data with Serial Correlation-A State-Space Approach R.H. Jones (1993)

- 48. Differential Geometry and Statistics M.K. Murray and J.W. Rice (1993)
- 49. Markov Models and Optimization M.H.A. Davis (1993)
- 50. Networks and Chaos—Statistical and Probabilistic Aspects O.E. Barndorff-Nielsen, J.L. Jensen and W.S. Kendall (1993)
- 51. Number-Theoretic Methods in Statistics K.-T. Fang and Y. Wang (1994)
- 52. Inference and Asymptotics O.E. Barndorff-Nielsen and D.R. Cox (1994)
- 53. Practical Risk Theory for Actuaries C.D. Daykin, T. Pentikäinen and M. Pesonen (1994)
- 54. Biplots J.C. Gower and D.J. Hand (1996)
- 55. Predictive Inference—An Introduction S. Geisser (1993)
- 56. Model-Free Curve Estimation M.E. Tarter and M.D. Lock (1993)
- 57. An Introduction to the Bootstrap B. Efron and R.J. Tibshirani (1993)
- 58. Nonparametric Regression and Generalized Linear Models P.J. Green and B.W. Silverman (1994)
- 59. Multidimensional Scaling T.F. Cox and M.A.A. Cox (1994)
- 60. Kernel Smoothing M.P. Wand and M.C. Jones (1995)
- 61. Statistics for Long Memory Processes J. Beran (1995)
- 62. Nonlinear Models for Repeated Measurement Data M. Davidian and D.M. Giltinan (1995)
- 63. Measurement Error in Nonlinear Models R.J. Carroll, D. Rupert and L.A. Stefanski (1995)
- 64. Analyzing and Modeling Rank Data J.J. Marden (1995)
- 65. Time Series Models—In Econometrics, Finance and Other Fields D.R. Cox, D.V. Hinkley and O.E. Barndorff-Nielsen (1996)
- 66. Local Polynomial Modeling and its Applications J. Fan and I. Gijbels (1996)
- 67. Multivariate Dependencies-Models, Analysis and Interpretation D.R. Cox and N. Wermuth (1996)
- 68. Statistical Inference-Based on the Likelihood A. Azzalini (1996)
- 69. Bayes and Empirical Bayes Methods for Data Analysis B.P. Carlin and T.A Louis (1996)
- 70. Hidden Markov and Other Models for Discrete-Valued Time Series I.L. MacDonald and W. Zucchini (1997)
- 71. Statistical Evidence—A Likelihood Paradigm R. Royall (1997)
- 72. Analysis of Incomplete Multivariate Data J.L. Schafer (1997)
- 73. Multivariate Models and Dependence Concepts H. Joe (1997)
- 74. Theory of Sample Surveys M.E. Thompson (1997)
- 75. Retrial Queues G. Falin and J.G.C. Templeton (1997)
- 76. Theory of Dispersion Models B. Jørgensen (1997)
- 77. Mixed Poisson Processes J. Grandell (1997)
- 78. Variance Components Estimation-Mixed Models, Methodologies and Applications P.S.R.S. Rao (1997)
- 79. Bayesian Methods for Finite Population Sampling G. Meeden and M. Ghosh (1997)
- Stochastic Geometry—Likelihood and computation O.E. Barndorff-Nielsen, W.S. Kendall and M.N.M. van Lieshout (1998)
- Computer-Assisted Analysis of Mixtures and Applications—Meta-Analysis, Disease Mapping and Others D. Böhning (1999)
- 82. Classification, 2nd edition A.D. Gordon (1999)
- 83. Semimartingales and their Statistical Inference B.L.S. Prakasa Rao (1999)
- 84. Statistical Aspects of BSE and vCJD-Models for Epidemics C.A. Donnelly and N.M. Ferguson (1999)
- 85. Set-Indexed Martingales G. Ivanoff and E. Merzbach (2000)
- 86. The Theory of the Design of Experiments D.R. Cox and N. Reid (2000)
- 87. Complex Stochastic Systems O.E. Barndorff-Nielsen, D.R. Cox and C. Klüppelberg (2001)
- 88. Multidimensional Scaling, 2nd edition T.F. Cox and M.A.A. Cox (2001)
- Algebraic Statistics Computational Commutative Algebra in Statistics G. Pistone, E. Riccomagno and H.P. Wynn (2001)
- 90. Analysis of Time Series Structure—SSA and Related Techniques N. Golyandina, V. Nekrutkin and A.A. Zhigljavsky (2001)
- 91. Subjective Probability Models for Lifetimes Fabio Spizzichino (2001)
- 92. Empirical Likelihood Art B. Owen (2001)
- 93. Statistics in the 21st Century Adrian E. Raftery, Martin A. Tanner, and Martin T. Wells (2001)
- Accelerated Life Models: Modeling and Statistical Analysis Vilijandas Bagdonavicius and Mikhail Nikulin (2001)

- 95. Subset Selection in Regression, Second Edition Alan Miller (2002)
- 96. Topics in Modelling of Clustered Data Marc Aerts, Helena Geys, Geert Molenberghs, and Louise M. Ryan (2002)
- 97. Components of Variance D.R. Cox and P.J. Solomon (2002)
- 98. Design and Analysis of Cross-Over Trials, 2nd Edition Byron Jones and Michael G. Kenward (2003)
- 99. Extreme Values in Finance, Telecommunications, and the Environment Bärbel Finkenstädt and Holger Rootzén (2003)
- 100. Statistical Inference and Simulation for Spatial Point Processes Jesper Møller and Rasmus Plenge Waagepetersen (2004)
- 101. Hierarchical Modeling and Analysis for Spatial Data Sudipto Banerjee, Bradley P. Carlin, and Alan E. Gelfand (2004)
- 102. Diagnostic Checks in Time Series Wai Keung Li (2004)
- 103. Stereology for Statisticians Adrian Baddeley and Eva B. Vedel Jensen (2004)
- 104. Gaussian Markov Random Fields: Theory and Applications Håvard Rue and Leonhard Held (2005)
- Measurement Error in Nonlinear Models: A Modern Perspective, Second Edition Raymond J. Carroll, David Ruppert, Leonard A. Stefanski, and Ciprian M. Crainiceanu (2006)
- 106. Generalized Linear Models with Random Effects: Unified Analysis via H-likelihood *Youngjo Lee, John A. Nelder, and Yudi Pawitan* (2006)
- 107. Statistical Methods for Spatio-Temporal Systems Bärbel Finkenstädt, Leonhard Held, and Valerie Isham (2007)
- 108. Nonlinear Time Series: Semiparametric and Nonparametric Methods Jiti Gao (2007)
- Missing Data in Longitudinal Studies: Strategies for Bayesian Modeling and Sensitivity Analysis Michael J. Daniels and Joseph W. Hogan (2008)
- Hidden Markov Models for Time Series: An Introduction Using R Walter Zucchini and Iain L. MacDonald (2009)
- 111. ROC Curves for Continuous Data Wojtek J. Krzanowski and David J. Hand (2009)
- 112. Antedependence Models for Longitudinal Data Dale L. Zimmerman and Vicente A. Núñez-Antón (2009)
- 113. Mixed Effects Models for Complex Data Lang Wu (2010)
- 114. Intoduction to Time Series Modeling Genshiro Kitagawa (2010)
- 115. Expansions and Asymptotics for Statistics Christopher G. Small (2010)
- 116. Statistical Inference: An Integrated Bayesian/Likelihood Approach Murray Aitkin (2010)
- 117. Circular and Linear Regression: Fitting Circles and Lines by Least Squares Nikolai Chernov (2010)
- 118. Simultaneous Inference in Regression Wei Liu (2010)
- 119. Robust Nonparametric Statistical Methods, Second Edition Thomas P. Hettmansperger and Joseph W. McKean (2011)
- 120. Statistical Inference: The Minimum Distance Approach Ayanendranath Basu, Hiroyuki Shioya, and Chanseok Park (2011)
- 121. Smoothing Splines: Methods and Applications Yuedong Wang (2011)
- 122. Extreme Value Methods with Applications to Finance Serguei Y. Novak (2012)
- 123. Dynamic Prediction in Clinical Survival Analysis Hans C. van Houwelingen and Hein Putter (2012)
- 124. Statistical Methods for Stochastic Differential Equations Mathieu Kessler, Alexander Lindner, and Michael Sørensen (2012)
- 125. Maximum Likelihood Estimation for Sample Surveys *R. L. Chambers, D. G. Steel, Suojin Wang, and A. H. Welsh* (2012)
- 126. Mean Field Simulation for Monte Carlo Integration Pierre Del Moral (2013)
- 127. Analysis of Variance for Functional Data Jin-Ting Zhang (2013)
- 128. Statistical Analysis of Spatial and Spatio-Temporal Point Patterns, Third Edition Peter J. Diggle (2013)
- 129. Constrained Principal Component Analysis and Related Techniques Yoshio Takane (2014)
- 130. Randomised Response-Adaptive Designs in Clinical Trials Anthony C. Atkinson and Atanu Biswas (2014)
- 131. Theory of Factorial Design: Single- and Multi-Stratum Experiments Ching-Shui Cheng (2014)
- 132. Quasi-Least Squares Regression Justine Shults and Joseph M. Hilbe (2014)
- Data Analysis and Approximate Models: Model Choice, Location-Scale, Analysis of Variance, Nonparametric Regression and Image Analysis Laurie Davies (2014)
- 134. Dependence Modeling with Copulas Harry Joe (2014)
- 135. Hierarchical Modeling and Analysis for Spatial Data, Second Edition Sudipto Banerjee, Bradley P. Carlin, and Alan E. Gelfand (2014)

- 136. Sequential Analysis: Hypothesis Testing and Changepoint Detection Alexander Tartakovsky, Igor Nikiforov, and Michèle Basseville (2015)
- 137. Robust Cluster Analysis and Variable Selection Gunter Ritter (2015)
- 138. Design and Analysis of Cross-Over Trials, Third Edition Byron Jones and Michael G. Kenward (2015)
- 139. Introduction to High-Dimensional Statistics Christophe Giraud (2015)
- 140. Pareto Distributions: Second Edition Barry C. Arnold (2015)

Pareto Distributions

Second Edition

Barry C. Arnold

University of California, Riverside Riverside, CA, USA



CRC Press is an imprint of the Taylor & Francis Group, an **informa** business A CHAPMAN & HALL BOOK CRC Press Taylor & Francis Group 6000 Broken Sound Parkway NW, Suite 300 Boca Raton, FL 33487-2742

© 2015 by Taylor & Francis Group, LLC CRC Press is an imprint of Taylor & Francis Group, an Informa business

No claim to original U.S. Government works Version Date: 20141216

International Standard Book Number-13: 978-1-4665-8485-3 (eBook - PDF)

This book contains information obtained from authentic and highly regarded sources. Reasonable efforts have been made to publish reliable data and information, but the author and publisher cannot assume responsibility for the validity of all materials or the consequences of their use. The authors and publishers have attempted to trace the copyright holders of all material reproduced in this publication and apologize to copyright holders if permission to publish in this form has not been obtained. If any copyright material has not been acknowledged please write and let us know so we may rectify in any future reprint.

Except as permitted under U.S. Copyright Law, no part of this book may be reprinted, reproduced, transmitted, or utilized in any form by any electronic, mechanical, or other means, now known or hereafter invented, including photocopying, microfilming, and recording, or in any information storage or retrieval system, without written permission from the publishers.

For permission to photocopy or use material electronically from this work, please access www.copyright.com (http://www.copyright.com/) or contact the Copyright Clearance Center, Inc. (CCC), 222 Rosewood Drive, Danvers, MA 01923, 978-750-8400. CCC is a not-for-profit organization that provides licenses and registration for a variety of users. For organizations that have been granted a photocopy license by the CCC, a separate system of payment has been arranged.

Trademark Notice: Product or corporate names may be trademarks or registered trademarks, and are used only for identification and explanation without intent to infringe.

Visit the Taylor & Francis Web site at http://www.taylorandfrancis.com

and the CRC Press Web site at http://www.crcpress.com

To my best friend, my wife, Carole, and to my youngest granddaughter, Kaelyn

Contents

Li	st of F	ìgures	xvii
Li	st of T	ables	xix
Pr	eface	to the First Edition	xxi
Preface to the Second Edition			
1	Histe	orical sketch with emphasis on income modeling	1
	1.1	Introduction	1
	1.2	The First Steps	2
	1.3	The Modern Era	7
2	Models for income distributions		
	2.1	What Is a Model?	19
	2.2	The Law of Proportional Effect (Gibrat)	20
	2.3	A Markov Chain Model (Champernowne)	21
	2.4	The Coin Shower (Ericson)	24
	2.5	An Open Population Model (Rutherford)	25
	2.6	The Yule Distribution (Simon)	27
	2.7	Income Determined by Inherited Wealth (Wold-Whittle)	30
	2.8	The Pyramid (Lydall)	30
	2.9	Competitive Bidding for Employment (Arnold and Laguna)	31
	2.10	Other Models	33
	2.11	Parametric Families for Fitting Income Distributions	35
3	Pare	to and related heavy-tailed distributions	41
	3.1	Introduction	41
	3.2	The Generalized Pareto Distributions	41
	3.3	Distributional Properties	47
		3.3.1 Modes	47
		3.3.2 Moments	47
		3.3.3 Transforms	49
		3.3.4 Standard Pareto Distribution	50
		3.3.5 Infinite Divisibility	50
		3.3.6 Reliability, $P(X_1 < X_2)$	50

	3.3.8	Products of Pareto Variables	52
		riouacto or runcto vurnacies	33
	3.3.9	Mixtures, Random Sums and Random Extrema	54
3.4	Order S	Statistics	55
	3.4.1	Ratios of Order Statistics	57
	3.4.2	Moments	59
	3.4.3	Moments in the Presence of Truncation	63
3.5	Record	l Values	64
3.6	Genera	lized Order Statistics	67
3.7	Residu	al Life	69
3.8	Asymp	ototic Results	71
	3.8.1	Order Statistics	71
	3.8.2	Convolutions	73
	3.8.3	Record Values	74
	3.8.4	Generalized Order Statistics	74
	3.8.5	Residual Life	75
	3.8.6	Geometric Minimization and Maximization	75
	3.8.7	Record Values Once More	77
3.9	Charac	terizations	78
	3.9.1	Mean Residual Life	78
	3.9.2	Truncation Equivalent to Rescaling	82
	3.9.3	Inequality Measures	83
	3.9.4	Under-reported Income	84
	3.9.5	Functions of Order Statistics	86
	3.9.6	Record Values	93
	3.9.7	Generalized Order Statistics	94
	3.9.8	Entropy Maximization	97
	3.9.9	Pareto (III) Characterizations	98
	3.9.10	Two More Characterizations	99
3.10	Related	d Distributions	100
3.11	The Di	screte Pareto (Zipf) Distribution	110
	3.11.1	Zeta Distribution	110
	3.11.2	Zipf Distributions	111
	3.11.3	Simon Distributions	112
	3.11.4	Characterizations	114
3.12	Remar	ks	115
Meas	sures of	f inequality	117
4.1	Apolog	gia for Prolixity	117
4.2	Comm	on Measures of Inequality of Distributions	118
	4.2.1	The Lorenz Curve	121
	4.2.2	Inequality Measures Derived from the Lorenz Curve	123
	4.2.3	The Effect of Grouping	135
	4.2.4	Multivariate Lorenz Curves	139
	4.2.5	Moment Distributions	145
	 3.4 3.5 3.6 3.7 3.8 3.9 3.9 3.10 3.11 3.12 Meat 4.1 4.2 	3.4 Order 8 3.4.1 3.4.2 3.4.3 3.5 Record 3.6 Genera 3.7 Residu 3.8 Asymp 3.8.1 3.8.2 3.8.3 3.8.4 3.8.5 3.8.6 3.8.7 3.9 Charac 3.9.1 3.9.2 3.9.3 3.9.4 3.9.2 3.9.3 3.9.4 3.9.5 3.9.6 3.9.7 3.9.8 3.9.9 3.9.10 3.10 Related 3.11 The Di 3.11.1 3.11.2 3.11.3 3.11.4 3.12 Remari Measures of 4.1 Apolog 4.2 Comm 4.2.1 4.2.2 4.2.3 4.2.4 4.2.5	 3.4 Order Statistics 3.4.1 Ratios of Order Statistics 3.4.2 Moments 3.4.3 Moments in the Presence of Truncation 3.5 Record Values 3.6 Generalized Order Statistics 3.7 Residual Life 3.8 Asymptotic Results 3.8.1 Order Statistics 3.8.2 Convolutions 3.8.3 Record Values 3.8.4 Generalized Order Statistics 3.8.5 Residual Life 3.8.6 Geometric Minimization and Maximization 3.8.7 Record Values Once More 3.9 Characterizations 3.9.1 Mean Residual Life 3.9.2 Truncation Equivalent to Rescaling 3.9.3 Inequality Measures 3.9.4 Under-reported Income 3.9.5 Functions of Order Statistics 3.9.6 Record Values 3.9.7 Generalized Order Statistics 3.9.8 Entropy Maximization 3.9.9 Pareto (III) Characterizations 3.9.10 Two More Characterizations 3.9.10 Two More Characterizations 3.9.11 The Discrete Pareto (Zipf) Distribution 3.11.1 Zeta Distributions 3.11.3 Simon Distributions 3.11.4 Characterizations 3.11.4 Characterizations 3.11.4 Characterizations 3.11.4 Characterizations 3.11.4 Characterizations 3.12 Remarks

				xiii
		4.2.6	Related Reliability Concepts	148
		4.2.7	Relations Between Inequality Measures	149
		4.2.8	Inequality Measures for Specific Distributions	149
		4.2.9	Families of Lorenz Curves	157
		4.2.10	Some Alternative Inequality Curves	166
	4.3	Inequal	ity Statistics	170
		4.3.1	Graphical Techniques	171
		4.3.2	Analytic Measures of Inequality	175
		4.3.3	The Sample Gini Index	183
		4.3.4	Sample Lorenz Curve	185
		4.3.5	Further Sample Measures of Inequality	189
		4.3.6	Relations Between Sample Inequality Measures	193
	4.4	Inequal	ity Principles and Utility	194
		4.4.1	Inequality Principles	195
		4.4.2	Transfers, Majorization and the Lorenz Order	196
		4.4.3	How Transformations Affect Inequality	203
		4.4.4	Weighting and Mixing	209
		4.4.5	Lorenz Order within Parametric Families	211
		4.4.6	The Lorenz Order and Order Statistics	212
		4.4.7	Related Orderings	214
		4.4.8	Multivariate Extensions of the Lorenz Order	216
	4.5	Optima	l Income Distributions	221
5	Infe	rence for	r Pareto distributions	223
	5.1	Introdu	ction	223
	5.2	Parame	ter Estimation	224
		5.2.1	Maximum Likelihood	224
		5.2.2	Best Unbiased and Related Estimates	227
		5.2.3	Moment and Quantile Estimates	233
		5.2.4	A Graphical Technique	236
		5.2.5	Bayes Estimates	236
		5.2.6	Bayes Estimates Based on Other Data Configurations	243
		5.2.7	Bayes Prediction	246
		5.2.8	Empirical Bayes Estimation	248
		5.2.9	Miscellaneous Bayesian Contributions	249
		5.2.10	Maximum Likelihood for Generalized Pareto Distributions	249
		5.2.11	Estimates Using the Method of Moments and Estimating	
			Equations for Generalized Pareto Distributions	254
		5.2.12	Order Statistic Estimates for Generalized Pareto Distribu-	
			tions	259
		5.2.13	Bayes Estimates for Generalized Pareto Distributions	263
	5.3	Interval	Estimates	264
	5.4	Parame	tric Hypotheses	268
	5.5	Tests to	Aid in Model Selection	269
	5.6	Special	ized Techniques for Various Data Configurations	275

•	
X1V	7

	5.7	Group	ed Data	283
	5.8	Inferen	nce for Related Distributions	288
		5.8.1	Zeta Distribution	289
		5.8.2	Simon Distributions	289
		5.8.3	Waring Distribution	292
		5.8.4	Under-reported Income Distributions	293
		5.8.5	Inference for Flexible Extensions of Pareto Models	296
		5.8.6	Back to Pareto	298
6	Mul	tivariat	te Pareto distributions	299
	6.0	Introd	uction	299
	6.1	A Hier	rarchy of Multivariate Pareto Models	299
		6.1.1	Mardia's First Multivariate Pareto Model	299
		6.1.2	A Hierarchy of Generalizations	300
		6.1.3	Distributional Properties of the Generalized Multivariate	
			Pareto Models	304
		6.1.4	Some Characterizations of Multivariate Pareto Models	309
	6.2	Altern	ative Multivariate Pareto Distributions	310
		6.2.1	Mixtures of Weibull Variables	310
		6.2.2	Transformed Exponential Variables	312
		6.2.3	Trivariate Reduction	313
		6.2.4	Geometric Minimization and Maximization	315
		6.2.5	Building Multivariate Pareto Models Using Independent	
			Gamma Distributed Components	319
		6.2.6	Other Bivariate and Multivariate Pareto Models	321
		6.2.7	General Classes of Bivariate Pareto Distributions	323
		6.2.8	A Flexible Multivariate Pareto Model	325
		6.2.9	Matrix-variate Pareto Distributions	327
	6.3	Relate	ed Multivariate Models	328
		6.3.1	Conditionally Specified Models	328
		6.3.2	Multivariate Hidden Truncation Models	332
		6.3.3	Beta Extensions	334
		6.3.4	Kumaraswamy Extensions	334
		6.3.5	Multivariate Semi-Pareto Distributions	335
	6.4	Pareto	and Semi-Pareto Processes	336
	6.5	Inferen	nce for Multivariate Pareto Distributions	340
		6.5.1	Estimation for Mardia's Multivariate Pareto Families	340
		6.5.2	Estimation for More General Multivariate Pareto Families	342
		6.5.3	A Confidence Interval Based on a Multivariate Pareto	
			Sample	347
		6.5.4	Remarks	349
	6.6	Multiv	variate Discrete Pareto (Zipf) Distributions	350

A Historical income data sources

		XV
B	Two representative data sets	359
С	A quarterly household income data set	369
References		

List of Figures

4.2.1	Minor concentration ratio.	133
4.2.2	The Lorenz zonoid for a bivariate Pareto (II) distribution with $\alpha = 9$	
	and parameters $(\mu_1, \mu_2, \sigma_1, \sigma_2) = (0, 0, 1, 1)$.	145
4.3.1	A Pareto chart for Indian income data (based on super tax records)	
	from Shirras (1935).	172
B .1	Pareto (II) model fitted to golfer data.	363
B.2	Standard Pareto model (left) and classical Pareto model (right), each	
	fitted to Texas counties data.	365
C 1		270
C.1	Pareto (II) model fitted to Mexican income data.	3/0