Recent Advances in Computational and Applied Mathematics

Theodore E. Simos
Editor

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Preface

This volume includes an exciting collection of papers on computational and applied mathematics presenting the recent advances in several areas of this field. All the papers have been peer reviewed by at least three reviewers.

In the paper entitled: “Fifty Years of Stiffness” by Luigi Brugnano, Francesca Mazzia and Donato Trigiante a review on the evolution of stiffness is presented. The authors also given a precise definition of stiffness which encompasses all the previous ones.

In the paper entitled: “Efficient Global Methods for the Numerical Solution of Nonlinear Systems of Two point Boundary Value Problems” by Jeff R. Cash and Francesca Mazzia, the authors investigated the numerical methods for the solution of nonlinear systems of two point boundary value problems in ordinary differential equations. More specifically they answer to the question: “which codes are currently available for solving these problems and which of these codes might we consider as being state of the art”. Finally the authors included some new codes for BVP’s which are written in MATLAB. These codes was not available before and allow us for the first time in the literature the possibility of comparing some important MATLAB codes for solving boundary value problems.

In the paper entitled: “Advances on collocation based numerical methods for Ordinary Differential Equations and Volterra Integral Equations” by D. Conte, R. D’Ambrosio, B. Paternoster a survey on collocation based numerical methods for the numerical integration of Ordinary Differential Equations and Volterra Integral Equations (VIEs) is presented. This survey starts from the classical collocation methods and arrive to the important modifications appeared in the literature. The authors consider also the multistep case and the usage of basis of functions other than polynomials.

In the paper entitled: “Basic Methods for Computing Special Functions” by Amparo Gil, Javier Segura and Nico M. Temme, the authors given a survey of methods for the numerical evaluation of special functions, that is, the functions that arise in many problems in the applied sciences. They considered a selection of basic methods which are used frequently in the numerical evaluation of special functions. They discussed also several other methods which are available. Finally, they given examples of recent software for special functions which use the above mentioned methods and they mentioned a list of new bibliography on computational aspects of special functions available on our website.

In the paper entitled: “Melt Spinning: Optimal Control and Stability Issue” by Thomas Gotz and Shyam S.N. Perera, the authors studied a mathematical model which describe the melt spinning process of polymer fibers. The authors used Newtonian and non-Newtonian models in order to describe the rheology of the polymeric material. They also investigated two important properties, the optimization and the stability of the process.

In the paper entitled: “On orthonormal polynomial solutions of the Riesz system in $\mathbb{R}^3$” by K. Gürlebeck and J. Morais, a special orthogonal system of polynomial solutions of the Riesz system in $\mathbb{R}^3$ is studied. This system presents a proportion with the complex case of the Fourier exponential functions $\{e^{in\theta}\}_{n\geq 0}$ on the unit circle and has the additional property that also the scalar parts of the polynomials form an orthogonal system. An application of the properties of the above system to the explicit calculation of conjugate harmonic functions with a certain regularity is also presented.

In the paper entitled: “Brief survey on the CP methods for the Schrödinger equation” by L.Gr. Ixaru, a review of the CP methods is presented. The authors investigated, after years of research in the subject all the advantages over other methods.

In the paper entitled: “Symplectic Partitioned Runge–Kutta methods for the numerical integration of periodic and oscillatory problems” by Z. Kalogiratou, Th. Monovasilis and T.E. Simos an investigation on Symplectic Partitioned Runge–Kutta methods (SPRK) is presented. More specifically they present the methodology for the construction of the exponentially/trigonometrically fitted SPRK. They applied the above methodology to methods with corresponding order up to fifth. The trigonometrically-fitted approach is based on two different types of construction: (i) fitting at each stage and (ii) Simos’s approach. The authors also derived SPRK methods with minimal phase-lag as well as phase-fitted SPRK methods. Finally, they applied the methods to several problems.

In the paper entitled: “On the Klein-Gordon equation on some examples of conformally flat spin 3-manifolds” by Rolf Sören Kraußhar a review about recent results on the analytic treatment of the Klein-Gordon equation on some conformally flat 3-tori and on 3-spheres is presented. The paper has two parts. In the first part the time independent Klein-Gordon equation $(\Delta - \alpha^2)u = 0$ $(\alpha \in \mathbb{R})$ on some conformally flat 3-tori associated with a representative system of conformally inequivalent spinor bundles is considered. In the second part a unified approach to represent the solutions to the Klein-Gordon equation on 3-spheres is described.

The *hp* version of the finite element method (*hp*-FEM) combined with adaptive mesh refinement is a particularly efficient method. For this method a single error estimate can not simultaneously determine whether it is better to do the refinement by h or by p. Several strategies for making this determination have been proposed over the years. In the paper entitled: “A Survey of *hp*-Adaptive Strategies for Elliptic Partial Differential Equations” by William F. Mitchell and Marjorie A. McClain, the authors studied these strategies and demonstrate the exponential convergence rates with two classic test problems.

In the paper entitled: "Vectorized Solution of ODEs in MATLAB with Control of Residual and Error" by L.F. Shampine a study on vectorization which is very important to the efficient computation in the popular problem-solving environment MATLAB is presented. More specifically, the author derived a new error control procedure which is based on vectorization. An explicit Runge—Kutta (7,8) pair of formulas that exploits vectorization is obtained. The new proposed method controls the local error at 8 points equally spaced in the span of a step. A new solver which is based on the above mentiobed pair and it is called `odevr7` is developed. This solver is much more efficient than the solver `ode45` which is recommended by MATLAB.

In the paper entitled: "Forecasting equations in complex-quaternionic setting" by W. Sprössig, the author considered classes of fluid flow problems under given initial value and boundary value conditions on the sphere and on ball shells in $\mathbb{R}^3$. The author interest is emphasized to the forecasting equations and the deduction of a suitable quaternionic operator calculus.

In the paper entitled: "Symplectic exponentially-fitted modified Runge–Kutta methods of the Gauss type: revisited" by G. Vanden Berghe and M. Van Daele, the development of symmetric and symplectic exponentially-fitted Runge–Kutta methods for the numerical integration of Hamiltonian systems with oscillatory solutions is studied. New integrators are obtained following the six-step procedure of Ixaru and Vanden Berghe (*Exponential Fitting*, Kluwer Academic, 2004).

We would like to express our gratitude to the numerous (anonymous) referees, to Prof. Hélène de Rode, the President of the European Academy of Sciences for giving us the opportunity to come up with this guest editorial work.

University of Peloponnese T.E. Simos

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Chapter 1
Fifty Years of Stiffness

Luigi Brugnano, Francesca Mazzia,
and Donato Trigiante

Abstract The notion of *stiffness*, which originated in several applications of a different nature, has dominated the activities related to the numerical treatment of differential problems for the last fifty years. Contrary to what usually happens in Mathematics, its definition has been, for a long time, not formally precise (actually, there are too many of them). Again, the needs of applications, especially those arising in the construction of robust and general purpose codes, require nowadays a formally precise definition. In this paper, we review the evolution of such a notion and we also provide a precise definition which encompasses all the previous ones.

Keywords Stiffness · ODE problems · Discrete problems · Initial value problems · Boundary value problems · Boundary value methods

Mathematics Subject Classification (2000) 65L05 · 65L10 · 65L99

Frustra fit per plura quod potest per pauciora.
Razor of W. of Ockham, doctor invincibilis.

Work developed within the project "Numerical methods and software for differential equations".

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1.1 Introduction

The struggle generated by the duality short times–long times is at the heart of human culture in almost all its aspects. Here are just a few examples to fix the idea:

- in historiography: Braudel's distinction among the geographic, social and individual times;[1]
- in the social sphere: Societies are organized according to three kinds of laws, i.e., codes (regulating short term relations), constitutions (regulating medium terms relations), and ethical laws (long term rules) often not explicitly stated but religiously accepted;
- in the economy sphere: the laws of this part of human activities are partially unknown at the moment. Some models (e.g., the Goodwin model [19]), permits us to say, by taking into account only a few variables, that the main evolution is periodic in time (and then predictable), although we are experiencing an excess of periodicity (chaotic behavior). Nevertheless, some experts claim (see, e.g., [18]) that the problems in the predictability of the economy are mainly due to a sort of gap in passing information from a generation to the next ones, i.e. to the conflict between short time and long time behaviors.[2]

Considering the importance of this concept, it would have been surprising if the duality "short times–long times" did not appear somewhere in Mathematics. As a matter of fact, this struggle not only appears in our field but it also has a name: *stiffness*.

Apart from a few early papers [10, 11], there is a general agreement in placing the date of the introduction of such problems in Mathematics to around 1960 [17]. They were the necessities of the applications to draw the attention of the mathematical community towards such problems, as the name itself testifies: "*they have been termed stiff since they correspond to tight coupling between the driver and the driven components in servo-mechanism*" ([12] quoting from [11]).

Both the number and the type of applications proposing difficult differential problems has increased exponentially in the last fifty years. In the early times, the problems proposed by applications were essentially initial value problems and, consequently, the definition of stiffness was clear enough and shared among the few experts, as the following three examples evidently show:

D1: *Systems containing very fast components as well as very slow components* (Dahlquist [12]).

D2: *They represent coupled physical systems having components varying with very different times scales: that is they are systems having some components varying much more rapidly than the others* (Liniger [31], translated from French).

---

[1] Moreover, his concept of *structure*, i.e. events which are able to accelerate the normal flow of time, is also interesting from our point of view, because it somehow recalls the mathematical concept of large variation in small intervals of time (see later).

[2] Even Finance makes the distinction between short time and long time traders.

D3: *A stiff system is one for which λ_{max} is enormous so that either the stability or the error bound or both can only be assured by unreasonable restrictions on h... Enormous means enormous relative to the scale which here is $\bar{t}$ (the integration interval)* ... (Miranker [34]).

The above definitions are rather informal, certainly very far from the precise definitions we are accustomed to in Mathematics, but, at least, they agree on a crucial point: the relation among stiffness and the appearance of different time-scales in the solutions (see also [24]).

Later on, the necessity to encompass new classes of difficult problems, such as Boundary Value Problems, Oscillating Problems, etc., has led either to weaken the definition or, more often, to define some consequence of the phenomenon instead of defining the phenomenon itself. In Lambert's book [29] five propositions about stiffness, each of them capturing some important aspects of it, are given. As matter of fact, it has been also stated that no universally accepted definition of stiffness exists [36].

There are, in the literature, other definitions based on other numerical difficulties, such as, for example, large Lipschitz constants or logarithmic norms [37], or non-normality of matrices [23]. Often is not even clear if stiffness refers to particular solutions (see, e.g. [25]) or to problems as a whole.

Sometimes one has the feeling that stiffness is becoming so broad to be nearly synonymous of difficult.

At the moment, even if the old intuitive definition relating stiffness to multiscale problems survives in most of the authors, the most successful definition seems to be the one based on particular effects of the phenomenon rather than on the phenomenon itself, such as, for example, the following almost equivalent items:

D4: *Stiff equations are equations where certain implicit methods ... perform better, usually tremendous better, than explicit ones* [11].

D5: *Stiff equations are problems for which explicit methods don't work* [21].

D6: *If a numerical method with a finite region of absolute stability, applied to a system with any initial condition, is forced to use in a certain interval of integration a step length which is excessively small in relation to the smoothness of the exact solution in that interval, then the system is said to be stiff in that interval* [29].

As usually happens, describing a phenomenon by means of its effects may not be enough to fully characterize the phenomenon itself. For example, saying that fire is what produces ash, would oblige firemen to wait for the end of a fire to see if the ash has been produced. In the same way, in order to recognize stiffness according to the previous definitions, it would be necessary to apply first one[3] explicit method and see if it works or not. Some authors, probably discouraged by the above defeats in giving a rigorous definition, have also affirmed that a rigorous mathematical definition of stiffness is not possible [20].

It is clear that this situation is unacceptable for at least two reasons:

[3] It is not clear if one is enough: in principle the definition may require to apply all of them.

- it is against the tradition of Mathematics, where objects under study have to be *precisely* defined;
- it is necessary to have the possibility to recognize *operatively* this class of problems, in order to increase the efficiency of the numerical codes to be used in applications.

Concerning the first item, our opinion is that, in order to gain in precision, it would be necessary to revise the concept of *stability* used in Numerical Analysis, which is somehow different from the homonym concept used in all the other fields of Mathematics, where stable are equilibrium points, equilibrium sets, reference solutions, etc., but not equations or problems[4] (see also [17] and [30]).

Concerning the second item, *operatively* is intended in the sense that the definition must be stated in terms of *numerically observable* quantities such as, for example, norms of vectors or matrices. It was believed that, seen from the applicative point of view, a formal definition of stiffness would not be strictly necessary: *Complete formality here is of little value to the scientist or engineer with a real problem to solve* [24].

Nowadays, after the great advance in the quality of numerical codes,[5] the usefulness of a formal definition is strongly recognized, also from the point of view of applications: *One of the major difficulties associated with the study of stiff differential systems is that a good mathematical definition of the concept of stiffness does not exist* [6].

In this paper, starting from ideas already partially exposed elsewhere [2, 4, 26], we will try to unravel the question of the definition of stiffness and show that a precise and operative definition of it, which encompasses all the known facets, is possible.

In order to be as clear as possible, we shall start with the simpler case of initial value for a single linear equation and gradually we shall consider more general cases and, eventually, we shall synthesize the results.

1.2 The Asymptotic Stability Case

For initial value problems for ODEs, the concept of stability concerns the behavior of a generic solution $y(t)$, in the neighborhood of a reference solution $\bar{y}(t)$, when the initial value is perturbed. When the problem is linear and homogeneous, the difference, $e(t) = y(t) - \bar{y}(t)$, satisfies the same equation as $\bar{y}(t)$. For nonlinear problems, one resorts to the linearized problem, described by the variational equation, which, essentially, provides valuable information only when $\bar{y}(t)$ is asymptotically stable. Such a variational equation can be used to generalize to nonlinear problems the arguments below which, for sake of simplicity, concerns only the linear case.

[4]Only in particular circumstances, for example in the linear case, it is sometimes allowed the language abuse: the nonlinear case may contain simultaneously stable and unstable solutions.

[5]A great deal of this improvement is due to the author of the previous sentence.

Originally, stiffness was almost always associated with initial value problems having asymptotically stable equilibrium points (dissipative problems) (see, e.g., Dahlquist [13]). We then start from this case, which is a very special one. Its peculiarities arise from the following two facts:[6]

- it is the most common in applications;
- there exists a powerful and fundamental theorem, usually called *Stability in the first approximation Theorem* or *Poincaré-Liapunov Theorem*, along with its corollary due to Perron[7], which allows us to reduce the study of stability of critical points, of a very large class of nonlinearities, to the study of the stability of the corresponding linearized problems (see, e.g., [9, 27, 35, 38]).

The former fact explains the pressure of applications for the treatment of such problems even before the computer age. The latter one provides, although not always explicitly recognized, the mathematical solid bases for the profitable and extensive use, in Numerical Analysis, of the linear test equation to study the fixed-h stability of numerical methods.

We shall consider explicitly the case where the linearized problem is autonomous, although the following definitions will take into account the more general case.

Our starting case will then be that of an initial value problem having an asymptotically stable reference solution, whose representative is, in the scalar case,

$$\begin{aligned} y' &= \lambda y, \quad t \in [0, T], \quad \mathrm{Re}\,\lambda < 0, \\ y(0) &= \eta, \end{aligned} \tag{1.2.1}$$

where the reference solution (an equilibrium point, in this case) has been placed at the origin. From what is said above, it turns out that it is not by chance that it coincides with the famous test equation.

Remark 1.1 It is worth observing that the above test equation is not less general than $y' = \lambda y + g(t)$, which very often appears in the definitions of stiffness: the only difference is the reference solution, which becomes $\bar{y}(t) = \int_0^t e^{\lambda(t-s)} g(s) ds$, but not the topology of solutions around it. This can be easily seen by introducing the new variable $z(t) = y(t) - \bar{y}(t)$ which satisfies exactly equation (1.2.1) and then, trivially, must share the same stiffness. Once the solution $z(t)$ of the homogeneous equation has been obtained, the solution $y(t)$ is obtained by adding to it $\bar{y}(t)$ which, in principle, could be obtained by means of a quadrature formula. This allows us to conclude that if any stiffness is in the problem, this must reside in the homogeneous part of it, i.e., in problem (1.2.1).

[6]We omit, for simplicity, the other fact which could affect new definitions, i.e., the fact that the solutions of the linear equation can be integrated over any large interval because of the equivalence, in this case, between asymptotic and exponential stability.

[7]It is interesting to observe that the same theorem is known as the *Ostrowsky's Theorem*, in the theory of iterative methods.

Remark 1.2 We call attention to the interval of integration $[0, T]$, which depends on our need for information about the solution, even if the latter exists for all values of t. This interval must be considered as datum of the problem. This has been sometimes overlooked, thus creating some confusion.

Having fixed problem (1.2.1), we now look for a mathematical tool which allows us to state formally the intuitive concept, shared by almost all the definitions of stiffness: i.e., we look for one or two parameters which tell us if in $[0, T]$ the solution varies rapidly or not. This can be done easily by introducing the following two measures for the solution of problem (1.2.1):

$$\kappa_c = \frac{1}{|\eta|} \max_{t\in[0,T]} |y(t)|, \qquad \gamma_c = \frac{1}{|\eta|} \frac{1}{T} \int_0^T |y(t)| dt, \tag{1.2.2}$$

which, in the present case, assume the values:

$$\kappa_c = 1, \qquad \gamma_c = \frac{1}{|\mathrm{Re}\,\lambda| T} (1 - e^{\mathrm{Re}\,\lambda T}) \approx \frac{1}{|\mathrm{Re}\,\lambda| T} = \frac{T^*}{T},$$

where $T^* = |\mathrm{Re}\,\lambda|^{-1}$ is the transient time. The two measures κ_c, γ_c are called *conditioning parameters* because they measure the sensitivity of the solution subject to a perturbation of the initial conditions in the infinity and in the l_1 norm.

Sometimes, it would be preferable to use a lower value of γ_c, i.e.,

$$\gamma_c = \frac{1}{|\lambda| T}. \tag{1.2.3}$$

This amounts to consider also the oscillating part of the solution (see also Remark 1.5 below).

By looking at Fig. 1.1, one realizes at once that a rapid variation of the solution in $[0, T]$ occurs when $k_c \gg \gamma_c$. It follows then that the parameter

$$\sigma_c = \frac{k_c}{\gamma_c} \equiv \frac{T}{T^*}, \tag{1.2.4}$$

which is the ratio between the two characteristic times of the problem, is more significant. Consequently, the definition of stiffness follows now trivially:

Definition 1.3 The initial value problem (1.2.1) is *stiff* if $\sigma_c \gg 1$.

The parameter σ_c is called *stiffness ratio*.

Remark 1.4 The width of the integration interval T plays a fundamental role in the definition. This is an important point: some authors, in fact, believe that stiffness should concern equations; some others believe that stiffness should concern problems, i.e., equations and data. We believe that both statements are partially correct: stiffness concerns equations, integration time, and a set of initial data (not a specific one of them). Since this point is more important in the non scalar case, it will be discussed in more detail later.

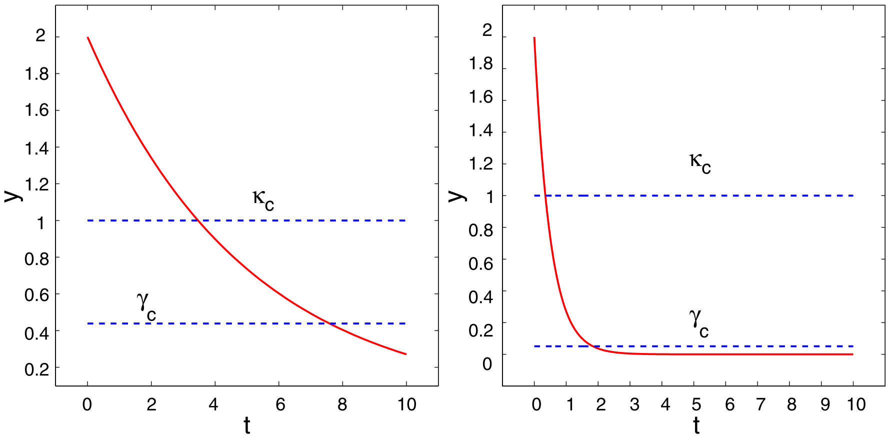

Fig. 1.1 Solutions and values of k_c and γ_c in the cases $\lambda = -0.2$ (*left plot*) and $\lambda = -2$ (*right plot*)

Remark 1.5 When γ_c is defined according to (1.2.3), the definition of stiffness continues to be also meaningful in the case $\mathrm{Re}\,\lambda = 0$, i.e., when the critical point is only marginally stable. In fact, let

$$\lambda = i\omega \equiv i\,\frac{2\pi}{T^*}.$$

Then,

$$\sigma_c = 2\pi\,\frac{T}{T^*},$$

and the definition encompasses also the case of *oscillating stiffness* introduced by some authors (e.g., [34]). Once again the stiffness is the ratio of two times. If information about the solution on the smaller time scale is needed, an adequately small stepsize should be used. It is worth noting that high oscillating systems (with respect to T) fall in the class of problems for which explicit methods do not work, and then are stiff according to definitions D4–D6.

When $\lambda = 0$, then $k_c = \gamma_c = \sigma_c = 1$.

In the case $\mathrm{Re}\,\lambda > 0$ (i.e., the case of an unstable critical point), both parameters k_c and γ_c grow exponentially with time. This implies that small variations in the initial conditions will imply exponentially large variations in the solutions, both pointwise and on average: i.e., the problem is *ill conditioned*.

Of course, the case $\mathrm{Re}\,\lambda = 0$ considered above cannot be considered as representative of more difficult nonlinear equations, since linearization is in general not allowed in such a case.

The linearization is not the only way to study nonlinear differential (or difference) equations. The so called *Liapunov second method* can be used as well (see, e.g., [22, 27, 38]). It has been used, in connection with stiffness in [5, 13–17], although not always explicitly named.[8] Anyway, no matter how the asymptotic stability of a reference solution is detected, the parameters (1.2.2) and Definition 1.3 continue to be valid. Later on, the problem of effectively estimating such parameters will also be discussed.

1.2.1 The Discrete Case

Before passing to the non scalar case, let us now consider the discrete case, where some interesting additional considerations can be made. Here, almost all we have said for the continuous case can be repeated. The first approximation theorem can be stated almost in the same terms as in the continuous case (see e.g. [28]).

Let the interval $[0, T]$ be partitioned into N subintervals of length $h_n > 0$, thus defining the mesh points: $t_n = \sum_{j=1}^{n} h_j$, $n = 0, 1, \ldots, N$.

The linearized autonomous problem is now:

$$y_{n+1} = \mu_n y_n, \quad n = 0, \ldots, N-1, \quad y_0 = \eta, \tag{1.2.5}$$

where the $\{\mu_n\}$ are complex parameters. The conditioning parameters for (1.2.5), along with the stiffness ratio, are defined as:

$$\begin{aligned} \kappa_d &= \frac{1}{|\eta|} \max_{i=0,\ldots,N} |y_i|, \qquad \gamma_d = \frac{1}{|\eta|} \frac{1}{T} \sum_{i=1}^{N} h_i \max(|y_i|, |y_{i-1}|), \\ \sigma_d &= \frac{k_d}{\gamma_d}. \end{aligned} \tag{1.2.6}$$

This permits us to define the notion of *well representation* of a continuous problem by means of a discrete one.

Definition 1.6 The problem (1.2.1) is *well represented* by (1.2.5) if

$$k_c \approx k_d, \tag{1.2.7}$$

$$\gamma_c \approx \gamma_d. \tag{1.2.8}$$

In the case of a constant mesh-size h, $\mu_n \equiv \mu$ and it easily follows that the condition (1.2.7) requires $|\mu| < 1$. It is not difficult to recognize the usual A-stability conditions for one-step methods (see Table 1.1). Furthermore, it is easily recognized that the request that condition (1.2.7) holds uniformly with respect to $h\lambda \in \mathbb{C}^-$ implies that the numerical method producing (1.2.5) must be implicit.

What does condition (1.2.8) require more? Of course, it measures how faithfully the integral $\int_0^T |y(t)| dt$ is approximated by the quadrature formula $\sum_{i=1}^{N} h_i \cdot \max(|y_i|, |y_{i-1}|)$, thus giving a sort of global information about the behavior of the

[8]Often, it appears under the name of one-sided Lipschitz condition.

Table 1.1 Condition (1.2.7) for some popular methods

Method	μ	Condition
Explicit Euler	$1+h\lambda$	$\lvert 1+h\lambda\rvert<1$
Implicit Euler	$\frac{1}{1-h\lambda}$	$\left\lvert\frac{1}{1-h\lambda}\right\rvert<1$
Trapezoidal rule	$\frac{1+h\lambda/2}{1-h\lambda/2}$	$\left\lvert\frac{1+h\lambda/2}{1-h\lambda/2}\right\rvert<1$

method producing the approximations $\{y_i\}$. One of the most efficient global strategies for changing the stepsize is based on monitoring this parameter [3, 4, 7, 8, 32, 33]. In addition to this, when finite precision arithmetic is used, then an interesting property of the parameter γ_d occurs [26]: if it is smaller than a suitably small threshold, this suggests that we are doing useless computations, since the machine precision has already been reached.

1.2.2 The non Scalar Case

In this case, the linearized problem to be considered is

$$y' = Ay, \quad t \in [0, T], \quad y(0) = \eta, \tag{1.2.9}$$

with $A \in \mathbb{R}^{m\times m}$ and having all its eigenvalues with negative real part. It is clear from what was said in the scalar case that, denoting by $\Phi(t) = e^{At}$ the fundamental matrix of the above equation, the straightforward generalization of the definition of the conditioning parameters (1.2.2) would lead to:

$$\kappa_c = \max_{t\in[0,T]} \|\Phi(t)\|, \qquad \gamma_c = \frac{1}{T}\int_0^T \|\Phi(t)\|dt, \qquad \sigma_c = \frac{\kappa_c}{\gamma_c}. \tag{1.2.10}$$

Indeed, these straight definitions *work most of the time*, as is confirmed by the following example, although, as we shall explain soon, not always.

Example 1.7 Let us consider the well-known Van der Pol's problem,

$$\begin{aligned} y_1' &= y_2, \\ y_2' &= -y_1 + \mu\, y_2(1-y_1^2), \quad t \in [0, 2\mu], \\ y(0) &= (2,\ 0)^T, \end{aligned} \tag{1.2.11}$$

whose solution approaches a limit cycle of period $T \approx 2\mu$. It is also very well-known that, the larger the parameter μ, the more difficult the problem is. In Fig. 1.2 we plot the parameter $\sigma_c(\mu)$ (as defined in (1.2.10)) for μ ranging from 0 to 10^3. Clearly, stiffness increases with μ.

Even though (1.2.10) works for this problem, this is not true in general. The problem is that the definition of stiffness as the ratio of two quantities may require a lower bound for the denominator. While the definition of κ_c remains unchanged, the definition of γ_c is more entangled. Actually, we need two different estimates of such a parameter:

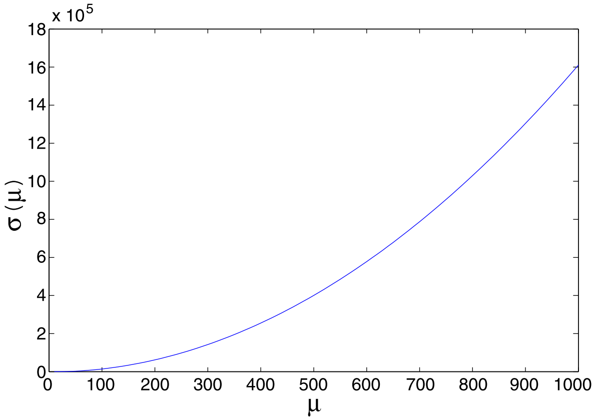

Fig. 1.2 Estimated stiffness ratio of Van der Pol's problem (1.2.11)

- an upper bound, to be used for estimating the conditioning of the problem in l_1 norm;
- a lower bound, to be used in defining σ_c and, then, the stiffness.

In the definition given in [2, 4], this distinction was not made, even though the definition was (qualitatively) completed by adding

$$\text{“}\textit{for at least one of the modes}\text{”}. \tag{1.2.12}$$

We shall be more precise in a moment. In the meanwhile, it is interesting to note that the clarification contained in (1.2.12) is already in one of the two definitions given by Miranker [34]:

A system of differential equations is said to be stiff on the interval $(0, \bar{t})$ *if there exists a solution of that system a component of which has a variation on that interval which is large compared to* $\frac{1}{\bar{t}}$,

where it should be stressed that the definition considers equations and not problems: this implies that the existence of largely variable components may appear for at least one choice of the initial conditions, not necessary for a specific one.

Later on, the definition was modified so as to translate into formulas the above quoted sentence (1.2.12). The following definitions were then given (see, e.g., [26]):

$$\begin{aligned}
\kappa_c(T,\eta) &= \frac{1}{\|\eta\|}\max_{0\le t\le T}\|y(t)\|, \qquad \kappa_c(T) = \max_{\eta}\kappa_c(T,\eta),\\
\gamma_c(T,\eta) &= \frac{1}{T\|\eta\|}\int_0^T \|y(t)\|dt, \qquad \gamma_c(T) = \max_{\eta}\gamma_c(T,\eta)
\end{aligned} \tag{1.2.13}$$

and

$$\sigma_c(T) = \max_{\eta}\frac{\kappa_c(T,\eta)}{\gamma_c(T,\eta)}. \tag{1.2.14}$$

The only major change regards the definition of σ_c. Let us be more clear on this point with an example, since it leads to a controversial question in the literature: i.e.,