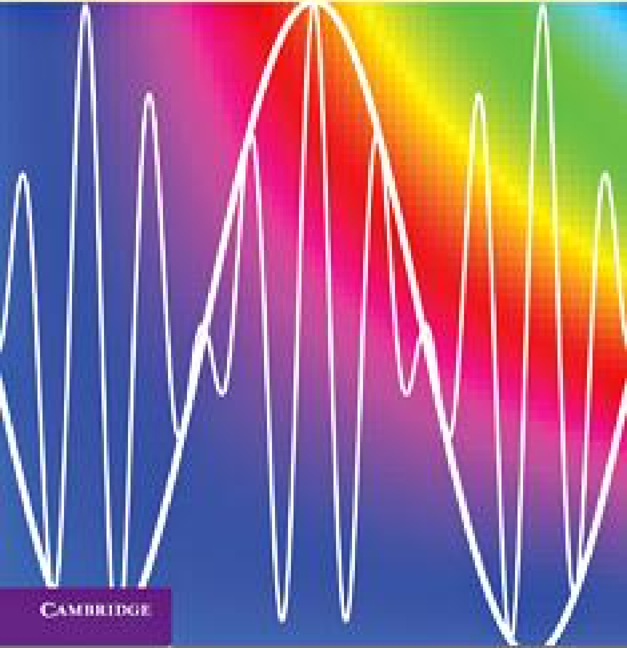
Sudhakar Nair

ADVANCED TOPICS IN APPLIED MATHEMATICS

For Engineering and the Physical Sciences



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Advanced Topics in Applied Mathematics

This book is ideal for engineering, physical science, and applied mathematics students and professionals who want to enhance their mathematical knowledge. *Advanced Topics in Applied Mathematics* covers four essential applied mathematics topics: Green's functions, integral equations, Fourier transforms, and Laplace transforms. Also included is a useful discussion of topics such as the Wiener-Hopf method, finite Hilbert transforms, Cagniard–De Hoop method, and the proper orthogonal decomposition. This book reflects Sudhakar Nair's long classroom experience and includes numerous examples of differential and integral equations from engineering and physics to illustrate the solution procedures. The text includes exercise sets at the end of each chapter and a solutions manual, which is available for instructors.

Sudhakar Nair is the Associate Dean for Academic Affairs of the Graduate College, Professor of Mechanical Engineering and Aerospace Engineering, and Professor of Applied Mathematics at the Illinois Institute of Technology in Chicago. He is a Fellow of the ASME, an Associate Fellow of the AIAA, and a member of the American Academy of Mechanics as well as Tau Beta Pi and Sigma Xi. Professor Nair is the author of numerous research articles and *Introduction to Continuum Mechanics* (2009).

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Illinois Institute of Technology

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Contents

Preface

This text is aimed at graduate students in engineering, physics, and applied mathematics. I have included four essential topics: Green's functions, integral equations, Fourier transforms, and Laplace transforms. As background material for understanding these topics, a course in complex variables with contour integration and analytic continuation and a second course in differential equations are assumed. One may point out that these topics are not all that advanced – the expected advanced-level knowledge of complex variables and a familiarity with the classical partial differential equations of physics may be used as a justification for the term "advanced." Most graduate students in engineering satisfy these prerequisites. Another aspect of this book that makes it "advanced" is the expected maturity of the students to handle the fast pace of the course. The fours topics covered in this book can be used for a one-semester course, as is done at the Illinois Institute of Technology (IIT). As an application-oriented course, I have included techniques with a number of examples at the expense of rigor. Materials for further reading are included to help students further their understanding in special areas of individual interest. With the advent of multiphysics computational software, the study of classical methods is in general on a decline, and this book is an attempt to optimize the time allotted in the curricula for applied mathematics.

I have included a selection of exercises at the end of each chapter for instructors to choose as weekly assignments. A solutions manual for these exercises is available on request. The problems are numbered in such a way as to simplify the assignment process, instead of clustering a number of similar problems under one number.

Classical books on integral transforms by Sneddon and on mathematical methods by Morse and Feshbach and by Courant and Hilbert form the foundation for this book. I have included sections on the Boundary Element Method and Proper Orthogonal Decomposition under integral equations – topics of interest to the current research community. The Cagniard–De Hoop method for inverting combined Fourier-Laplace transforms is well known to researchers in the area of elastic waves, and I feel it deserves exposure to applied mathematicians in general. Discrete Fourier transform leading to the fast Fourier algorithm and the Z-transform are included.

I am grateful to my numerous students who have read my notes and corrected me over the years. My thanks also go to my colleagues, who helped to proofread the manuscript, Kevin Cassel, Dietmar Rempfer, Warren Edelstein, Fred Hickernell, Jeff Duan, and Greg Fasshauer, who have been persistent in instilling applied mathematics to believers and nonbelievers at IIT, and, especially, for training the students who take my course. I am also indebted to my late colleague, Professor L. N. Tao, who shared the applied mathematics teaching with me for more than twenty-five years.

The editorial assistance provided by Peter Gordon and Sara Black is appreciated.

The Mathematica™ package from Wolfram Research was used to generate the number function plots.

My wife, Celeste, has provided constant encouragement throughout the preparation of the manuscript, and I am always thankful to her.

1

GREEN'S FUNCTIONS

Before we introduce the **Green's functions**, it is necessary to familiarize ourselves with the idea of **generalized functions** or **distributions**. These are called generalized functions as they do not conform to the definition of functions. They are often unbounded and discontinuous. They are characterized by their integral properties as linear functionals.

1.1 HEAVISIDE STEP FUNCTION

Although this is a simple discontinuous function (not a generalized function), the **Heaviside step function** is a good starting point to introduce generalized functions. It is defined as

$$h(x) = \begin{cases} 0, & x < 0, \\ 1/2, & x = 0, \\ 1, & x > 0. \end{cases} \tag{1.1}$$

The value of the function at $x = 0$ is seldom needed as we always approach the point $x = 0$ either from the right or from the left (see Fig. 1.1). When we consider representation of this function using, say, Fourier series, the series converges to the mean of the right and left limits if there is a discontinuity. Thus, $h(0) = 1/2$ will be the converged result for such a series.

Using the Heaviside function, we can express the **signum** function, which has a value of 1 when the argument is positive, and a value of -1

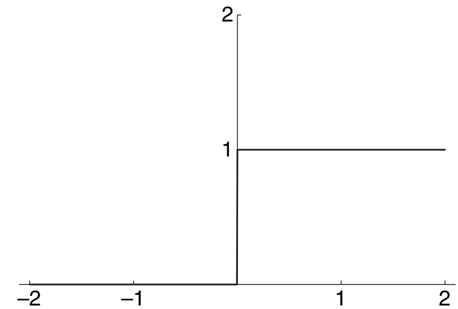

Figure 1.1. Heaviside step function.

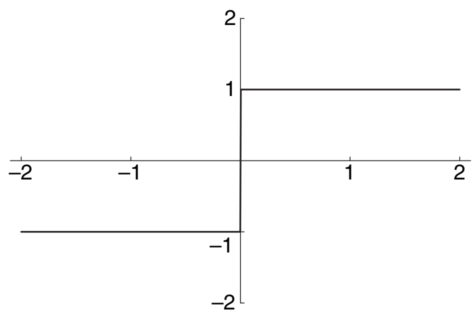

Figure 1.2. Signum function sgn(x).

when the argument is negative (see Fig. 1.2), as

$$\operatorname{sgn}(x) = 2h(x) - 1. \tag{1.2}$$

We may convert an even function of x to an odd function simply by multiplying by sgn(x).

The function shown in Fig. 1.3 can be written as

$$f(x) = h(a - |x|). \tag{1.3}$$

This is known as the **Haar function**, which plays an important role in image processing as a basis for **wavelet** expansions. In wavelet analysis,

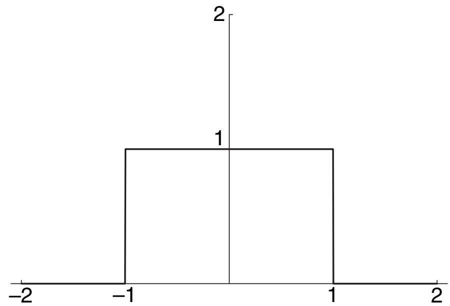

Figure 1.3. Haar function ($a = 1$).

families of Haar functions with support a, $a/2$, $a/4, \ldots, a/2^n$ are used as a basis to represent functions.

1.2 DIRAC DELTA FUNCTION

The Dirac delta function has its origin in the idea of *concentrated* charges in electromagnetics and quantum mechanics. In mechanics, the Dirac delta $\delta(x)$ is useful in representing concentrated forces. We can view this generalized function as the derivative of the Heaviside function, which is zero everywhere except at the origin. At the origin it is infinity. As a consequence, its integral from $-\epsilon$ to $+\epsilon$ is unity. As is the case for all generalized functions, we consider the delta function as the limit of various sequences of functions. For example, consider the sequence of functions shown in Fig. 1.4, which depends on the parameter ϵ,

$$f(x;\epsilon) = \begin{cases} 0, & |x| > \epsilon, \\ \frac{1}{2\epsilon}, & |x| < \epsilon. \end{cases} \tag{1.4}$$

In the limit $\epsilon \to 0, f(x;\epsilon) \to \delta(x)$.

Note that

$$\int_{-\infty}^{\infty} f(x;\epsilon)\,dx = \int_{-\epsilon}^{\epsilon} f(x;\epsilon)\,dx = 1 = \int_{-\infty}^{\infty} \delta(x)\,dx = \int_{-\epsilon}^{\epsilon} \delta(x)\,dx. \tag{1.5}$$

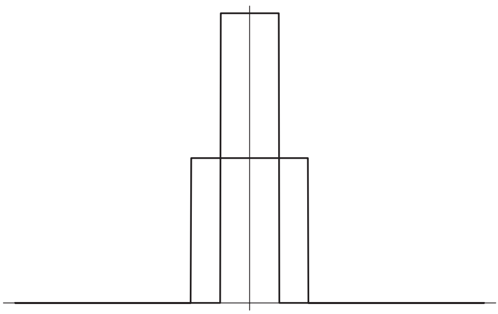

Figure 1.4. A delta sequence using Haar functions.

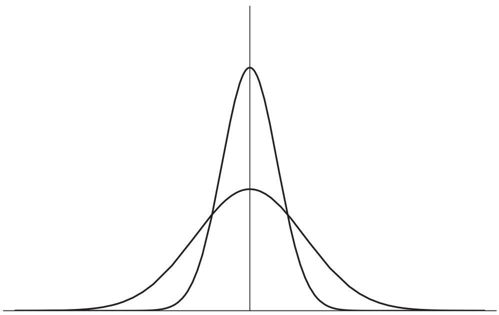

Figure 1.5. Another delta sequence using probability functions.

Another sequence of continuous functions which forms a delta sequence is given (see Fig. 1.5) by the Gauss functions or probability functions:

$$f(x;n) = \frac{n}{\sqrt{\pi}} e^{-n^2x^2}. \tag{1.6}$$

We can see that the area under this curve remains unity for all values of n. Let

$$I = \int_{-\infty}^{\infty} ne^{-n^2x^2}\,dx = \int_{-\infty}^{\infty} e^{-x^2}\,dx, \tag{1.7}$$

where we substituted $nx \to x$. Using polar coordinates,

$$\begin{aligned}
I^2 &= \int_{-\infty}^{\infty} e^{-x^2}\,dx \int_{-\infty}^{\infty} e^{-y^2}\,dy \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\,dxdy \\
&= \int_0^{\infty}\int_0^{2\pi} e^{-r^2}\, r\,drd\theta \\
&= 2\pi \int_0^{\infty} e^{-r^2}\, r\,dr = -\pi e^{-r^2}\Big|_0^{\infty} = \pi .
\end{aligned} \tag{1.8}$$

We frequently encounter the integral I, which has the value

$$I = \int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}. \tag{1.9}$$

As $n \to \infty, f(x;n) \to \delta(x)$.

By shifting the origin from $x = 0$ to $x = \xi$, we can move the spike of the delta function to the point ξ. This new function has the properties,

$$\delta(x-\xi) = 0, \quad x \neq \xi, \tag{1.10}$$

$$\int_{\xi-\epsilon}^{\xi+\epsilon} \delta(x-\xi)\,dx = 1. \tag{1.11}$$

An important property of the delta function is localization under integration. As usual, properties of the generalized functions are proved using the corresponding sequences. For any smooth function $\phi(x)$, which is nonzero only in a finite interval (a,b), using the sequence (1.4), we have

$$\begin{aligned}
&\int_{-\infty}^{\infty} \phi(x)\delta(x-\xi)\,dx \\
&\qquad = \lim_{\epsilon\to 0}\int_{\xi-\epsilon}^{\xi+\epsilon} \phi(x)\,\frac{dx}{2\epsilon} \\
&\qquad = \lim_{\epsilon\to 0}\int_{\xi-\epsilon}^{\xi+\epsilon} \left[\phi(\xi) + \phi'(\xi)(x-\xi) + \frac{1}{2}\phi''(\xi)(x-\xi)^2 + \cdots\right]\frac{dx}{2\epsilon}
\end{aligned}$$

$$
\begin{aligned}
&= \lim_{\epsilon \to 0} \left[\phi(\xi) + \frac{1}{2}\phi'(\xi)\epsilon + \frac{1}{6}\phi''(\xi)\epsilon^2 + \cdots \right] \\
&= \phi(\xi). \qquad (1.12)
\end{aligned}
$$

Integrals involving a *scaled* delta function can be evaluated as shown:

$$
\begin{aligned}
\int_{-\infty}^{\infty} \phi(x)\delta(\frac{x-\xi}{a})\, dx &= \int_{-\infty}^{\infty} \phi(ax')\delta(x'-\xi')\, adx' \\
&= a\phi(\xi), \qquad (1.13)
\end{aligned}
$$

where we used $x' = x/a$, $\xi' = \xi/a$, $a > 0$.

1.2.1 Macaulay Brackets

A simplified notation to represent integrals of the δ function was introduced in the context of structural mechanics by Macaulay. In this notation

$$\delta(x-\xi) = \langle x-\xi\rangle^{-1}, \qquad (1.14)$$

$$h(x-\xi) = \langle x-\xi\rangle^{0}, \qquad (1.15)$$

$$\int \langle x-\xi\rangle^{n} dx = \frac{1}{n+1}\langle x-\xi\rangle^{n+1}, \quad n \neq -1, \qquad (1.16)$$

$$\int \langle x-\xi\rangle^{-1} dx = \langle x-\xi\rangle^{0}. \qquad (1.17)$$

All of these functions are zero when the quantity inside the brackets is negative. For $n < 0$, some books omit the factor $1/(n+1)$ in the integral. We may include higher derivatives of the delta function in this group. In one-dimensional problems, such as the deflection of beams under concentrated loads, this notation is useful.

1.2.2 Higher Dimensions

In an n-dimensional Euclidian space $\mathbf{R}^n$ with coordinates $(x_1,x_2,\ldots,x_n)$, we use the simplified notation for the infinitesimal volume,

$$dx_1 dx_2 \ldots dx_n = d\boldsymbol{x}, \tag{1.18}$$

and the same for functions

$$\phi(x_1,x_2,\ldots,x_n) = \phi(\boldsymbol{x}), \quad \delta(x_1,x_2,\ldots,x_n) = \delta(\boldsymbol{x}). \tag{1.19}$$

Then the n-dimensional integral,

$$\int_{\mathbf{R}^n} \phi(\boldsymbol{x})\delta(\boldsymbol{x})\, d\boldsymbol{x} = \phi(\mathbf{0}). \tag{1.20}$$

More often we encounter situations involving two and three-dimensional spaces and cartesian coordinates (x,y) or (x,y,z), and the above result directly applies. When we use polar coordinates (or spherical coordinates) the appropriate area element (or volume element)

$$dA = r\,dr\,d\theta \quad (\text{or} \quad dV = r^2 \sin^2\phi\, dr\, d\phi\, d\theta) \tag{1.21}$$

is used. For example,

$$\int_0^\infty \int_0^{2\pi} f(r,\theta)\delta(r-r_0,\theta-\theta_0) r\,dr\,d\theta = f(r_0,\theta_0) \tag{1.22}$$

1.2.3 Test Functions, Linear Functionals, and Distributions

We conclude this section by introducing the idea of generalized functions or distributions as linear functionals over test functions.

A function, $\phi(x)$, is called a test function if (a) $\phi \in \mathcal{C}^\infty$, (b) it has a closed bounded (compact) support, and (c) ϕ and all of its derivatives decrease to zero faster than any power of $|x|^{-1}$.

A linear functional $\mathcal{T}$ of ϕ maps it into a scalar. This is done using an integral over $-\infty$ to ∞ as an inner product with some other sequence or distribution, f. If we denote this mapping as

$$\mathcal{T}_f[\phi] = \int_{-\infty}^{\infty} f(x)\phi(x)\,dx, \tag{1.23}$$

then the δ-distribution is defined by the relation

$$\mathcal{T}_\delta[\phi] = \int_{-\infty}^{\infty} \delta(x)\phi(x)\,dx = \phi(0). \tag{1.24}$$

A sequence $\delta_n (n = 0, 1, \ldots, \infty)$ converges to the δ-function if

$$\lim_{n\to\infty} \mathcal{T}_{\delta_n}[\phi] \to \phi(0). \tag{1.25}$$

A distribution $\mu(x)$ is the derivative of the δ-distribution if

$$\mathcal{T}_\mu[\phi] = -\phi'(0), \tag{1.26}$$

as

$$\int_{-\infty}^{\infty} \delta'(x)\phi(x)\,dx = -\int_{-\infty}^{\infty} \delta(x)\phi'(x)\,dx = -\phi'(0). \tag{1.27}$$

This way, we can define higher-order derivatives of the delta function. In engineering, concentrated forces, charges, fluid flow sources, vortex lines, and the like are represented using delta functions. The delta function is also called a unit impulse function in control theory.

1.2.4 Examples: Delta Function

Using the property, for any test function ϕ,

$$\int_{-\infty}^{\infty} \phi(x)\psi(x)\,dx = \phi(\xi) \tag{1.28}$$

implies

$$\psi(x) = \delta(x - \xi), \tag{1.29}$$

prove that, for $\alpha, \beta \neq 0$,

(a)

$$\frac{\partial}{\partial\alpha}\delta(\alpha x) = -\frac{1}{\alpha^2}\delta(x), \tag{1.30}$$