Therefore, if we have s summations symbols in the 1st set of values, the general theorem is

$$|a_{gh}| = \sum_r |a_{uv}|$$

where
$$\begin{aligned} g &= 1,2,\ldots\ldots m \\ h &= m+1, m+2,\ldots\ldots.2m \\ u &= 1,2,\ldots\ldots\ldots m-s, r_1, r_2,\ldots.r_s. \\ v &= m+1, m+2,\ldots.r_1-1, m-s+1, r_1+1,\ldots.r_2-1, m-s+ \\ &\quad 2,\ldots.r_s-1, m, r_s+1,..2m \\ &\quad r_1, r_2\ldots\ldots r_s = m+1, m+2,\ldots\ldots\ldots 2m. \end{aligned}$$

The number of determinants in the summation on the right equals ${}^mC_s$. Hence we see that Kronecker's Theorem is a special case of the theorem stated here—the case where $s = 1$.

[1] We wish to express our indebtedness to Professor F. D. Murnaghan for suggesting the possibility of extending this relation, after reading Professor E. B. Stouffer's paper on this subject in these PROCEEDINGS, January, 1926.

[2] Murnaghan, F. D., *Amer. Math. Monthly*, **32**, May, 1925 (233–241). Murnaghan, F. D., *Bull. of Amer. Math. Soc.*, **31**, July, 1925 (323–329).

ON THE CLASSICAL DIRICHLET PROBLEM FOR GENERAL DOMAINS

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1. *Introduction.*—We shall be concerned with the most general *domains*, or open continua, in space of three dimensions. It will be convenient to denote domains, and points in them, by capital letters, and to denote their boundaries, and points of these boundaries, by corresponding small letters. We shall consider explicitly only domains, T, whose boundaries, t, are bounded point-sets, a restriction which can be removed by inversions.

A function $F(p)$, defined on t, is said to be continuous if to every $\epsilon > 0$, there corresponds a $\delta > 0$, such that $|F(p) - F(q)| < \epsilon$ for any two points of t whose distance, pq, $< \delta$. By the classical Dirichlet problem, we mean that of determining a function, $U(P)$, harmonic in T, such that $U(P) \longrightarrow F(p)$ as $P \longrightarrow p$. If T is not bounded, the additional restriction is placed on $U(P)$ that it shall vanish at infinity like the potential of bounded charges.

The problem is not always possible. Zaremba[1] and Lebesgue[2] have exhibited cases in which exceptional boundary points exist, at which functions having otherwise the properties of solutions cannot approach the assigned boundary values. But in any case a function $U(P)$, bounded and harmonic in T, can be associated with given continuous boundary values, $F(p)$, which will approach $F(p)$ at every regular (not exceptional) boundary point.[3] Wiener[4] derived important properties of this function $U(P)$, which we shall call the *sequence solution* of the Dirichlet problem.

We shall establish the results of Wiener in a way simpler than has heretofore been given, taking as point of departure a generalized function of Green for T, and shall obtain additional light on the sequence solution. We shall then give certain theorems on the *capacity* of point-sets, and show, in particular, that sets of 0 capacity are the most general at which harmonic functions can have removable singularities. Finally, we show that if the exceptional points of t form a set of 0 capacity, the sequence solution is the only function which is bounded and harmonic in T and which approaches $F(p)$ at all regular boundary points.[5]

2. *Green's Function.*—Let $[T_n]$ denote a nested set of *normal* domains with T as limit; i.e., a set such that the classical Dirichlet problem is possible for each T_n and any continuous boundary values, such that T_n is contained in all the following domains of the set, and such that each point of T is contained in some T_n. Then, by the reasoning of Harnack,[6] the functions of Green, $G_n(P,Q)$, for the domains, T_n, and fixed pole, Q, form a sequence which approaches a limit uniformly in any closed sub-region of T omitting Q. This limit, $G(P,Q)$, we shall call Green's function for T, leaving open the question as to whether it vanishes at all boundary points or not.

From the manner of its formation, $G(P,Q)$ is seen to have the following properties. $G(P,Q) - 1/PQ$ is harmonic in T, save for a possible removable singularity at Q; if T is not bounded, $G(P,Q)$ vanishes regularly at infinity; $G(Q,P) = G(P,Q)$.

THEOREM I. *$G(P,Q)$ is independent of the particular set of nested domains, $[T_n]$, used in its definition.* For, if $[T'_n]$ denote a second set, $G(P,Q)$ dominates all the functions of Green for the domains T'_n, and hence their limit, $G'(P, Q)$, i.e., $G'(P,Q) \leqq G(P,Q)$. Similarly, $G(P,Q) \leqq G'(P,Q)$, and the theorem follows.

If T is a normal domain, $G(P,Q) \longrightarrow 0$ as $P \longrightarrow p$, p being any boundary point. The converse was first proved by Lichtenstein.[6] If T is not normal, there will be boundary points at which $G(P,Q)$ does not approach 0. We define as *regular* and *exceptional* boundary points, those at which $G(P,Q)$ does, or does not, approach 0, respectively.

THEOREM II. *The definition of regular and exceptional points is independent of the position of the pole, Q.* Let Q and Q' be two points of T, and let R be a region consisting of a domain containing Q and Q', together with its boundary, r, and lying in T. Let m be the positive minimum of $G(P,Q')$ on r, and M the maximum of $1/PQ$ on r. As $G_n(P,Q) \leqq 1/PQ$, it follows that $G_n(P,Q) \leqq (M/m)G(P,Q')$ on r, and since this inequality holds on t_n, the boundary of T_n, it holds also throughout $T_n - R$. Hence the limiting form of this inequality, $G(P,Q) \leqq (M/m)G(P,Q')$ holds in $T-R$, and $G(P,Q)$ must approach 0 at any boundary point at which $G(P,Q')$ does.

The above reasoning, with the inequality given, leads at once to

LEMMA I. *Let Q be confined to a closed sub-region, R, of T. Then*

(*a*) *as P approaches a regular boundary point, $G(P,Q) \longrightarrow 0$ uniformly as to Q*, and

(*b*) *if p is exceptional, but such that for fixed Q' a sequence $[P_k]$ with p as limit point exists for which $G(P_k,Q') \longrightarrow 0$, $G(P_k,Q) \longrightarrow 0$, uniformly as to Q, as $P \longrightarrow p$.*[7]

3. *The Sequence Solution.*—The sequence solution is derived from any given continuous boundary values, $F(p)$, just as was Green's function for the boundary values $1/PQ$. To $F(p)$ there correspond infinitely many functions, continuous in $T + t$, and coinciding with $F(p)$ on t. Let $F(P)$ be such a function. Then, to each of an infinite sequence of nested normal domains, $[T_n]$, there corresponds a function, $u_n(P)$, harmonic in T_n, and coinciding with $F(P)$ on t_n. Wiener's theorem is that the sequence $[u_n(P)]$ approaches a limit, $U(P)$, uniformly in any closed sub-region of T, and that $U(P)$ is independent of the particular extension of $F(p)$ to the points of T, and of the set $[T_n]$ employed. In calling $U(P)$ the sequence solution of the Dirichlet problem for T and $F(p)$, we are, of course, using the word solution in an extended sense, for a strict solution is not always possible.

We turn now to a study of the sequence solution, starting with the case in which $F(p)$ are the boundary values of a polynomial, $F(P)$, in the cartesian coördinates of the point P of T. We also assume first that T is bounded. The Laplacian of $F(P)$ is again a polynomial,

$$\nabla^2 F(P) = f(P). \tag{1}$$

We form the function

$$\phi(P) = \lim_{n \longrightarrow \infty} \phi_n(P), \qquad \phi_n(P) = \frac{1}{4\pi} \iiint_{T_n} f(Q) G(P,Q) dV_Q. \tag{2}$$

In case T_n fails to have Jordan content, the integral above is to be understood as the limit of the integral over a set of nested domains with content, and with T_n as limit. As $f(P)$ is a polynominal, and as $G(P,Q) \leqq 1/PQ$, the sequence $[\phi_n(P)]$ is seen to be uniformly convergent in any closed subregion of T, and its limit, $\phi(P)$ to be independent of the particular set $[T_n]$ used.

THEOREM III. *The function* $U(P) = F(P) + \phi(P)$ *is bounded and harmonic in* T, *and approaches the boundary values* $F(p)$ *at every regular boundary point.*

$U(P)$ is obviously bounded. If, in the integral for $\phi_n(P)$, $G(P,Q)$ be written $1/PQ + g(P,Q)$, the Laplacian of the first integral thus arising is well known to be $-4\pi f(P)$, while that of the second is 0. Hence $\nabla^2\phi_n(P) = -f(P)$, and so $\nabla^2\phi(P) = -f(P)$ because the sequence $[\phi_{n+m}(P) - \phi_n(P)]$ is, for fixed n, uniformly convergent in T_n, and its terms are harmonic in T_n. Thus, as $\nabla^2\phi(P) = -f(P)$, and $\nabla^2 F(P) = f(P)$, $\nabla^2 U(P) = 0$.

It remains to show that $U(P) \longrightarrow F(p)$, that is, that $\phi(P) \longrightarrow 0$, as $P \longrightarrow p$, a regular boundary point. Since $\phi(P)$ is independent of $[T_n]$, we may assume the boundary of each T_n to be interior to all following domains of the set. Then, for any fixed n, and any $\epsilon > 0$, there is a $\delta > 0$, such that when $Pp < \delta$, $G(P,Q) < \epsilon$ for all Q in T_n. Hence, by lemma I, if M is the upper limit of $f(P)$ in T, and if B is the volume of a cube containing T,

$$|\phi_n(P)| \leqq \frac{MB}{4\pi}\epsilon \tag{3}$$

for $Pp < \delta$. On the other hand, for positive integral m,

$$|\phi_{n+m}(P) - \phi_n(P)| \leqq \frac{M}{4\pi}\iiint\limits_{T_{n+m}-T_n} G(P,Q)dV_Q \leqq \frac{M}{4\pi}\iiint\limits_{T_{n+m}-T_n} \frac{1}{PQ}dV_Q,$$

and the last integral, extended over a region of given volume is least when the region is a sphere with P as center. Hence

$$|\phi_{n+m}(P) - \phi_n(P)| \leqq \frac{M}{8\pi}\left(\frac{3V}{4\pi}\right)^{2/3}, \tag{4}$$

where V is the inner content of T less the volume of T_n. For large n, V, and therefore $|\phi_{n+m}(P) - \phi_n(P)|$, is arbitrarily small, independently of m and P. Then, with n fixed, the right-hand member of (3) may be made arbitrarily small by sufficiently restricting Pp, and hence the same is true for $\phi_{n+m}(P)$, independently of m. The same is, therefore, true of $\phi(P)$, that is, $\phi(P) \longrightarrow 0$ as $P \longrightarrow p$.

If p is an exceptional boundary point, but one for which a sequence $[P_k]$, as described in lemma I (*b*), exists, then by the same reasoning, $\phi(P_k) \longrightarrow O$ as $P_k \longrightarrow p$. Thus the sequence solution approaches the given boundary values on certain sequences of points even at exceptional points, provided these sequences of points exist. And they do exist, save at certain boundary points at which possible discontinuities of the solution are removable (p. 404).

We have assumed, in the proof of theorem III, that T was bounded. If this is not the case, but t is bounded, say, by a sphere with center O and radius R, we proceed as follows, still assuming that $F(p)$ are the boundary values of a polynomial, $F_1(P)$. We form a function, $H(P)$, as follows: when $r = OP \leqq R$, $H(P) = 1$; when $r \geqq 2R$, $H(P) = 0$, and when $R \leqq r \leqq 2R$, $H(P)$ is a polynomial of degree 5 in r, so chosen that $H(P)$ and its first two derivatives with respect to r are continuous. The continuous extension of $F(p)$ to the points of T is then $F(P) = F_1(P)H(P)$. Then $\nabla^2 F(P)$ is bounded, and identically 0 for $r \geqq 2R$. The integrals are then again taken over bounded domains.

THEOREM IV. *The function $U(P)$ of the preceding theorem is the sequence solution of the Dirichlet problem for T and $F(p)$.* This will follow at once if it is clear that $G_n(P,Q)$ may replace $G(P,Q)$ in the formula (2) without changing the limit, for then $\phi_n(P) + F(P)$ is exactly the sequence function $u_n(P)$. The replacement in question, however, does not affect the limit, because of the uniform convergence of $[G_n(P,Q)]$ to $G(P,Q)$ in any closed sub-region of T, and of the fact that $G_n(P,Q) \leqq G(P,Q) \leqq 1/PQ$. Thus the sequence solution exists, and is independent of $[T_n]$, and approaches the given boundary values in the ways described above, all on the assumption of polynomial boundary values—a restriction which we now remove.

Let $F(p)$ be any continuous function, and let $F(P)$ be a function, continuous in $T + t$, and coinciding with $F(p)$ on t. Let $F'(P)$ be a polynomial which differs from $F(P)$ in $T + t$ by less than ϵ, and let $u_n(P)$ and $u_n'(P)$ be the sequence functions for the same set $[T_n]$ and with the values $F(P)$ and $F'(P)$ on t_n. Then $u_n'(P) - \epsilon \leqq u_n(P) \leqq u_n'(P) + \epsilon$ in T_n. As $[u_n'(P)]$ approaches a limit, $U'(P)$, uniformly in any closed sub-region, R, of T, for all great enough n, and all P in R, $U'(P) - 2\epsilon \leqq u_n(P) \leqq U'(P) + 2\epsilon$. And as there is a relation of this form for every positive ϵ, $[u_n(P)]$ approaches a limit, $U(P)$, uniformly in R. Furthermore, the limiting forms of the last inequalities show that the upper and lower limits of $U(P)$ at each regular boundary point lie between $F(p) - 3\epsilon$ and $F(p) + 3\epsilon$. But as these limits are independent of ϵ they coincide and $U(p) \longrightarrow F(p)$. The situation is similar in the case of sequence approach at certain exceptional points.

Finally, the sequence solution is independent of the continuous extension of $F(p)$ to the points of T. If $F(P)$ and $K(P)$ are two such extensions, and $[u_n(P)]$ and $[v_n(P)]$ the corresponding sequences, the sequence $[u_n(P) - v_n(P)]$ whose terms coincide with $F(P) - K(P)$ on t_n approaches 0 uniformly, since $F(P) - K(P)$ approaches 0 uniformly at the points of t. Hence the sequence solutions $U(P)$ and $V(P)$ corresponding to $F(P)$ and $K(P)$ coincide.

If, in the above use of approximating polynomials, T is not bounded, we may take any continuous extension, $F(P)$, of the given boundary values, and, using the function $H(P)$, previously established, approximate to $F(P)H(P)$ by a polynomial in the portion of T within a sphere about O

of radius $2R$. Instead of this polynomial, we then use its product by $H(P)$ for $r < 2R$ and 0 for $r \geqq 2R$.

4. *The Capacity of Point-Sets.*—It is possible, as Wiener[8] has shown, to extend the notion of the capacity of a conductor to any bounded point-set. We shall give his definition, in slightly modified form, and give some properties and applications of capacity.

Let B be any bounded point-set, and let B' be the set consisting of B and its limit points. B' may bound a number of domains, but a part of B', which we denote by t, will bound a domain, T, extending to infinity. Consider the sequence solution, $v(P)$, of the Dirichlet problem for T and the boundary values 1 on t. We call $v(P)$ the *conductor potential* of the set B. The *capacity* of B is then the total charge of which $v(P)$ is the potential, or the Gauss integral, $\frac{1}{4\pi}\int\int \frac{\partial v}{\partial n}\,dS$, extended over any smooth surface containing B in its interior, n being the normal to this surface, directed inward.

THEOREM V. *If A is a part of B, the capacity of A is less than, or equal to, that of B.* The continuous extension of the boundary values may be taken as 1 itself, and the nested regions, $[S_n]$ and $[T_n]$, used in constructing the conductor potentials $w(P)$ and $v(P)$ of A and B, respectively, may be so chosen that S_n is always a part of T_n. Then, on t_n, $v_n(P) = 1$, while $w_n(P) \leqq 1$. Hence $v(P) - w(P)$ is never negative. If $v(P) - w(P)$ vanishes at any point of T, it vanishes identically, and the capacities of A and B are then equal. Otherwise, let R denote an equipotential surface $v(P) - w(P) = C$, C being small enough to insure this surface being the smooth boundary of a simply connected domain. On R, $\frac{\partial}{\partial n}[v(P) - w(P)]$ cannot be negative, since if it were, $v(P) - w(P)$ would exceed C at some point outside R. Hence the Gauss integral over R cannot be negative, and the capacity of A is, therefore, less than, or equal to, that of B.

THEOREM VI. *Let T denote the infinite domain bounded by a part of B and its limit points, and $[T_n]$ a nested set of normal domains with T as limit. Then the capacities of the boundaries, t_n, of T_n, converge, decreasing monotonely, to the capacity of B.* This follows from theorem V, and the uniform convergence in any closed sub-region of T of the sequence $[v_n(P)]$ defining the conductor potential of B, together with the uniform convergence of the sequence of normal derivatives of $v_n(P)$ to the corresponding derivatives of $v(P)$.

LEMMA II. *If B is a bounded set of capacity $c > 0$, the value of the conductor potential of B at a point, P, of T, is subject to the inequalities*

$$\frac{c}{p'P} \leqq v(P) \leqq \frac{c}{p''P},$$

where p' and p'' are the points of B' (B and its limit points) *farthest from, and nearest to P, respectively.* Because of theorem VI, and the independence of the sequence solution, $v(P)$, of the nested set, $[T_n]$, it will suffice if the lemma is proved on the assumption that B consists of a finite number of surfaces with continuous principal curvatures. Then $v(P)$ is the potential of a continuous distribution of a charge, c, on B, and $v(P)$ is decreased by concentrating the charge at the point of B farthest from P, and increased by concentrating it at the nearest point.

COROLLARY. *If B is a bounded set of capacity 0, its conductor potential vanishes identically in T.* For B is enclosable in the boundary, t_n, of a normal domain, T_n, the capacity of t_n being arbitrarily small. Thus the conductor potential of B is dominated at P by an arbitrarily small number, that is, it is 0.

5. *Removable Singularities of Harmonic Functions.*—A function which is bounded and harmonic in the neighborhood of a point, or of an arc of a smooth curve, can be so defined at such points as to be harmonic there also. That is, such points or arcs can be the seat only of removable singularities. A complete generalization of this fact is possible by means of the notion of capacity.

THEOREM VII. *Let T be any domain, and let B be any part of its boundary with the following properties: (a) the set $T' = T + B$ is still a domain, and (b) the capacity of B is 0. Then any bounded function, harmonic in T, may be so defined on B as to be harmonic in T'.*[9] To show this, we first establish the following lemma.

LEMMA III. *Let σ be the surface of a sphere, and $F(p)$ a function of p on σ, bounded, and continuous except at the points of a set, s, of capacity 0. Then there exists a function, $U(P)$, harmonic and bounded within σ, and approaching the boundary values $F(p)$ at all points where this function is continuous.* We assume $0 \leqq F(p) \leqq 1$. The proof can then be extended to the more general case by a linear transformation. We assume also that s is closed. Its capacity is not affected by adding its limit points. By theorem VI, for every positive integer n, there exists a finite number of smooth surfaces containing s in their interior, and of capacity less than $1/n$. Within these surfaces, a finite number of smooth curves can be drawn on σ, bounding a domain, s_n, on σ, which contains s, and whose capacity is less than $1/n$, by theorem V. Furthermore, these domains s_n can be chosen so that each includes all the following.

If we define $F'(p) = 0$ on s_n, and $F'(p) = F(p)$ elsewhere on σ, the discontinuities of $F'(p)$ are limited to a finite number of smooth curves, so that Poisson's integral defines a function $u_n(P)$, harmonic within σ, and approaching the boundary values $F'(p)$ save on the curves bounding s_n. The sequence $[u_n(P)]$ is never decreasing, and so approaches a harmonic limit, $U(P)$, uniformly in any closed region within σ. The bounds of

$U(P)$ are clearly 0 and 1. It remains to show that $U(P) \longrightarrow F(p)$ as $P \longrightarrow p$, a point of continuity of $F(p)$. Let R be any closed region of space containing no point of s. Then, given $\epsilon > O$, n can be taken so large that the conductor potential, $v_n(P)$, of s_n, is less than ϵ in R, by theorem VI. But on $\sigma - s_n$, $u_{n+m}(P) - u_n(P) = 0$, for any positive integer m, and this difference never exceeds 1. Hence $-v_n(P) \leqq u_{n+m}(P) - u_n(P) \leqq v_n(P)$ on σ, and hence also in its interior.[10] Hence the sequence $[u_n(P)]$ converges uniformly in the closed region consisting of the points of R within or on σ, and as the terms of the sequence have the required boundary values there, so does their limit.

Returning now to the theorem, let Q be any point of B. Since T' is a domain, Q is the center of some sphere, σ, lying entirely in T'. Let $V(P)$ be any function, bounded and harmonic in T. Then its values on σ define a function $F(p)$ satisfying the conditions of the lemma, by theorem V. If $U(P)$ be the function, harmonic in σ, determined by these values $F(p)$ in accordance with the lemma, $V(P) - U(P)$ is bounded within and on σ, is harmonic within σ except possibly at the points of B, and has the boundary values O on σ except at the points of B. If the bound of the absolute value of $V(P) - U(P)$ is M, this difference, and its negative, is dominated by the conductor potential of any set of surfaces, t_n, enclosing B. Hence as the capacity of B is O, $V(P) - U(P) = 0$ throughout the interior of σ except on B. Hence if $V(P)$ is defined on the portion of B within σ as equal to $U(P)$, it will be harmonic within σ. Thus $V(P)$ may be so defined at all points of B as to be harmonic in T'.

THEOREM VIII. *Conversely, if B has the property (a) of theorem VII, and if any function bounded and harmonic in T can have only removable singularities at the points of B, then B has capacity 0.* For the conductor potential of B is bounded and harmonic in T, and so has only removable singularities. When properly defined on B, it is harmonic everywhere, and vanishes at infinity, and so vanishes identically. Hence the capacity of B is 0.

From theorem VII, it follows that boundary sets of the type B can have no influence on the classical Dirichlet problem. They are called *improper sets* by Bouligand, who defines them as sets of points at which $\underline{lim}\ G(P,Q) > 0$. It is not obvious that the concepts are equivalent, so we give a proof.

LEMMA IV. *The conductor potential of any set of positive capacity has 1 as upper limit.* For if its upper limit were $M < 1$, then, $v(P)$ denoting this conductor potential, $v(P)/M$ would be dominated by all the terms of the sequence $[v_n(P)]$ whose limit defines $v(P)$, and we should have $v(P) \geqq v(P)/M$, a contradiction.

Now let b be the set of points at which $\underline{lim}\ G(P,Q) > 0$. The set $T' = T + b$ is an open continuum, since the points of b are limit points of T, and since if a point of b were not an interior point of T', it would have to be the limit of a set of points at which $\overline{lim}\, G(P,Q) = 0$ and so could not itself be a point at which $\underline{lim}\, G(P,Q) > 0$. We wish to prove that b has capacity 0. Let A be a point of b, and σ a sphere with center at A and lying in T'. Then, within and on σ, $G(P,Q)$ has a positive lower bound, m. Otherwise there would be in this closed region a point at which $\overline{lim}\, G(P,Q) = 0$, i.e., a point not in T'. Let s denote the set of points of b in σ, with their limit points. Then s has capacity 0. Otherwise, the conductor potential of s would have the upper limit 1. Now on a sphere σ', of radius r and center Q, lying in T, this conductor potential, $v(P)$, has a maximum, $M < 1$, so that $\frac{1}{r}\left(\frac{1-v(P)}{1-M}\right)$ is a function harmonic in T, greater than, or equal to $1/r$ on σ', and never negative. It therefore dominates the sequence functions $G_n(P,Q)$ in $T-\sigma'$, and hence also $G(P,Q)$. But it has the lower limit 0 in σ, whereas that of $G(P,Q)$ is $m > 0$. Thus the assumption that the capacity of s is positive has led to a contradiction.

It is now possible to remove from the boundary of T a portion of b, contained in s but containing all the points of b in a sphere about A, so as still to leave an open continuum in place of T. The set removed, being a part of s, has capacity 0. Theorem VII then shows that any function, harmonic and bounded in T may be so defined on this portion of b as to be harmonic at A. But as A was any point of b, the harmonic function may be defined at all points of b so as to be harmonic in $T' = T + b$. Hence by theorem VIII, b is of capacity 0, and so of the type of the set B.

Conversely, if P is any point of a set of type B, $\underline{lim}\, G(P,Q) > 0$ at P. For, after the definition of $G(P,Q)$ on B has been properly made, $G(P,Q)$ is harmonic at P, and so continuous and positive.

We close with a uniqueness theorem.

THEOREM IX. *If the exceptional boundary points of T form a set of capacity 0, there is one and only one function, $U(P)$, bounded and harmonic in T, which approaches preassigned continuous boundary values, $F(p)$, at every regular boundary point.* One such function is the sequence solution. If there were two, their difference, $V(P)$, would be bounded and harmonic in T, and approach O at every regular boundary point. If M is a bound for $|V(P)|$, $-V(P)/M$ and $V(P)/M$ are dominated by the conductor potential of any set including the exceptional points of t, and so, by a now familiar argument, vanish.[11]

[1] Zaremba, *Acta Math.*, **34** (1911), p. 310.

[2] Lebesgue, *C. R. des Sce. de la Soc. Math. de France* (1913), p. 17.

[3] Kellogg, *Proc. Amer. Acad.*, **58**, No. 14 (1923), p. 528.

[4] "Certain Notions in Potential Theory," *Jour. of Math. and Phys. of the Mass. Inst. of Tech.*, **3** (1924), p. 25.

[5] Results closely parallel to a number of those to be given have recently been published by Bouligand in two stimulating memoirs, *Mémorial des sci. math.*, Paris, 1926, and *Annales de la Soc. Polonaise de Math.* (1925), pp. 59–112. Of these, some had